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THE SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY BY MINIMIZING THE BOUNDARY RESIDUAL IN THE SPACE OF BIHARMONIC FUNCTIONS†

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The mixed problem of the two-dimensional theory of elasticity is reduced to a biharmonic boundary-value problem for the stress function (this is only taken in the case when the forces are specified on the whole boundary), to solve which a numerical–analytic approach is proposed based on approximation by biharmonic functions. It enables the geometrical dimension of the boundary-value problem to be reduced, thereby reducing it to minimization of the boundary residual. The approximate solution obtained satisfies identically all the equations of the theory of elasticity, and the specified boundary conditions are approximated with high accuracy.

Related methods of solving problems of mechanics were considered in [1–8].

1. THE GENERAL SCHEME OF THE METHOD

The idea of using harmonic basis functions to solve the Dirichlet problem by a variational method [1] can be extended [7] to arbitrary boundary-value problems for systems of linear partial differential equations

$$\mathbf{Af} = \mathbf{0} \ \text{ in } G, \quad \mathbf{Bf} = \mathbf{g} \ \text{ on } \Gamma \tag{1.1}$$

where G is a region in $\mathbb{R}^k$ with boundary Γ, $\mathbf{f}$: $G \to \mathbb{R}^\alpha$ is the required function (in general vector-valued), A is a linear differential operator in the linear space $V(G)$ of functions from G into $\mathbb{R}^\alpha$, $\mathbf{g}$: $\Gamma \to \mathbb{R}^\beta$ is a given function which specifies the boundary values for $\mathbf{f}$ and combinations of its derivatives, $V(\Gamma)$ is a certain space of boundary functions from Γ into $\mathbb{R}^\beta$ and B is a linear differential operator from $V(G)$ into $V(\Gamma)$, which defines the nature of the boundary conditions. In this section when we refer to the "solution" of an equation or a boundary-value problem we shall have in mind either a regular solution or a generalized (weak) solution.

The sequence of approximate solutions of boundary-value problem (1.1) is constructed in the form

$$\mathbf{f}_N(x) = \sum_1^N c_i \varphi_i(x) \tag{1.2}$$

where $c_i \in \mathbb{R}$, $\{\varphi_i(x) \mid x \in G, i = 1, 2, \dots\}$ is a certain linearly independent system of functions, complete in the sense defined below, in the set $K = K(A, G, B)$ of all solutions of the equation $\mathbf{Af} = \mathbf{0}$ in G lying in the domain of the operator B (K is a linear space over $\mathbb{R}$, because **A** and **B** are linear). By virtue of the choice of the approximating basis functions $\varphi_i(X)$ and the linearity of the operator **A** $\mathbf{Af}_N = \mathbf{0}$ in G for $\forall c_i \in \mathbb{R}$. Hence, the coefficients c_i are chosen so that the approximate solution (1.2) suits the given boundary conditions more accurately, i.e. c_i are determined through minimizing the functional of the form

$$F[\mathbf{f}] \equiv F_g[\mathbf{f}] = \|\mathbf{Bf} - \mathbf{g}\|_\Gamma^2, \quad \mathbf{f} \in K(A, G, B) \tag{1.3}$$

in the space K. Here $\|\cdot\|_\Gamma$ denotes a real function on $V(\Gamma)$ which vanishes only on the zero element of the space $V(\Gamma)$ (these functions will be called "norms" for brevity, bearing in mind that all the norms

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possess this property). It ensures that the set of minimal elements of the functional (1.3) in K coincides with the set of the solutions of boundary-value problem (1.1) with fixed function g (both coincide with the set $\{\mathbf{f} \in K \mid F_g|\mathbf{f}| = 0\}$).

By its definition the space K is identical with the set of all solutions of boundary-value problem (1.1) for different $\mathbf{g} \in V(\Gamma)$. Hence, the class of functions $\mathbf{g} \in V(\Gamma)$, for which boundary-value problem (1.1) is solvable, coincides with the subspace $B(K)$. Henceforth, we will assume that the solution $\hat{\mathbf{f}}$ of boundary-value problem (1.1) exists and is unique, i.e. the functional (1.3) has a single minimal element in K. To find it we can use any of the existing methods for minimizing functionals. If, due to a proper choice of the "norm", the restriction $F_N(c_1, \ldots, c_N)$ of the functional F onto N-dimensional subspace of functions of the form (1.2) turns out to be a differentiable function, the necessary minimum conditions $\partial F_N/\partial c_k = 0$ ($k = 1, \ldots, N$) give the system of algebraic equations for determining the coefficients c_i. These equations are linear when the function F_N is quadratic (for example, if the norm is generated by any scalar product in $V(\Gamma)$).

The functional F (the measure of the deviation of the boundary values of the approximate solution from the specified boundary conditions) can be chosen differently using various "norms" in $V(\Gamma)$. Both the form of the system of equations for c_i (and of course, the approximate solution (1.2)), and the type of convergence of the boundary residual $\mathbf{Rf}_N \equiv \mathbf{Bf}_N - \mathbf{g}$ to zero as $N \to \infty$ and the corresponding type of convergence of the approximations $\mathbf{f}_N$ to the exact solution $\hat{\mathbf{f}}$ of problem (1.1), depend on the form of the functional F. The choice of the functional F determines the required form of completeness of the family of basis functions φ_i in the space K, which ensures the construction of a minimizing sequence $\mathbf{f}_N$ of F, i.e. the convergence of $F[\mathbf{f}_N] = \|\mathbf{Rf}_N\|_\Gamma$ to $\min_{f\in K} F\{\mathbf{f}\} = F[\hat{\mathbf{f}}] = 0$ as $N \to \infty$ and, finally, the convergence of $\mathbf{f}_N$ to $\hat{\mathbf{f}}$ for any $\mathbf{g} \in \mathbf{B}(K)$ in (1.1). The sequence $\{\mathbf{f}_N\}$ of the functions (1.2) with coefficients c_i, obtained through minimizing the functional (1.3) in the subspace of all linear combinations of the form (1.2), will obviously be a minimizing sequence for F for any $\mathbf{g} \in \mathbf{B}(K)$ if and only if the family of basis functions $\{\varphi_i\}$ is F-complete (with respect to the functional (1.3)) in the space K, i.e. for any $\mathbf{g} \in \mathbf{B}(K)$ and $\varepsilon > 0$ the numbers $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ exist such that

$$F\left[\sum_1^k \alpha_i \varphi_i\right] \equiv \left\| \sum_1^k \alpha_i \mathbf{B}\varphi_i - \mathbf{g} \right\|_\Gamma^2 < \varepsilon$$

Since $\mathbf{A}(\mathbf{f}_N - \hat{\mathbf{f}}) = \mathbf{0}$ in G, we can draw a conclusion from the convergence of $\|\mathbf{Bf}_N - \mathbf{g}\|_\Gamma \to 0$ regarding the behaviour of the error $\mathbf{f}_N - \hat{\mathbf{f}}$ inside G as $N \to \infty$, using integral representations of the solutions of partial differential equations and the analogy of the maximum principle for harmonic functions: if the norm $\|\cdot\|_G$ in K, corresponding to the desired type of convergence of $\mathbf{f}_N \to \hat{\mathbf{f}}$ agrees with $\|\cdot\|_\Gamma$ in $B(K)$ so that

$$0 \leqslant \|\mathbf{f}\|_G \leqslant M\|\mathbf{Bf}\|_\Gamma, \quad \forall \mathbf{f} \in K \tag{1.4}$$

where $M > 0$ is independent of $\mathbf{f}$, then $\|\mathbf{Bf}_N - \mathbf{g}\|_\Gamma \to 0$ as $N \to \infty$ yields $\|\mathbf{f}_N - \hat{\mathbf{f}}\|_G \to 0$: $\|\mathbf{f}_N - \hat{\mathbf{f}}\|_G \leqslant M\|\mathbf{B}(\mathbf{f}_N - \hat{\mathbf{f}})\|_\Gamma = M\|\mathbf{Bf}_N - \mathbf{B}\ \hat{\mathbf{f}}\|_\Gamma = M\|\mathbf{Bf}_N - \mathbf{g}\|_\Gamma$. If the solution of problem (1.1) is unique for all $\mathbf{g} \in \mathbf{B}(K)$, the operator $\mathbf{B}$: $K \to \mathbf{B}(K)$ is bijective. Inequality (1.4) indicates that the inverse operator $\mathbf{B}^{-1}$: $\mathbf{B}(K) \to K$ is continuous with respect to the chosen norms, i.e. boundary-value problem (1.1) is well posed.

The main difference between this approach and the majority of numerical methods is the fact that initially it is ensured that approximation (1.2) belongs to the set of solutions of the differential equation of problem (1.1), common for all boundary-value problems which describe a certain physical phenomenon, and only then the individuality of each specific problem is taken into account when satisfying the boundary conditions approximately. As a consequence of this it becomes possible: (i) to reduce the geometrical dimensions of the problem, (ii) to use a universal system of basis functions (for fixed A in (1.1)) to solve problems which differ in the boundary conditions, (iii) to obtain a smooth approximate solutions which satisfy the differential equations of the problem identically, (iv) to prove the convergence of the sequence of approximate solutions using estimates for the solutions of the differential equations inside the region in terms of the norm of their boundary values (similar to the maximum principle), fundamental solutions, integral representations of the solutions, and imbedding theorems, (v) to develop a closed semi-analytic highly accurate computational algorithm with a minimum amount of input data (only analytic expressions which specify the boundary conditions and the parametric representation of the boundary of the region and several controlling numerical parameter-modes are to be entered); (vi) to estimate the error of the approximate solution inside the region via the deviation of its boundary values from the specified boundary conditions, simply and reliably monitor the authenticity of the numerical calculations; (vii) to calculate the required quantities not only at the nodes, but any point of the region, use analytic differentiation to determine the additional physical quantities, related to the required quantities; (viii) to describe analytically in the approximate solution the singularities of the exact solution which arise at corner points of the boundary and caused by discontinuities of the boundary data, and the application of concentrated forces (sources). These features enable the class of test problems used to analyse the errors of other numerical methods to be extended considerably. The sphere of application of the method at the initial stage of development includes boundary-value problems for linear partial differential equations with constant coefficients, when one can construct a family of solutions of the equations suitable for use as basis functions, for example, hyperbolic–trigonometric or polynomial [4, 5, 7]. In particular, it includes mixed boundary-value problems of the mechanics of piecewise-homogeneous anisotropic media. Problems for non-linear equations with variable coefficients can also be reduced to the solution of a sequence of such problems.

2. THE REDUCTION OF A MIXED PROBLEM OF THE TWO-DIMENSIONAL THEORY OF ELASTICITY TO A BOUNDARY-VALUE PROBLEM FOR THE BIHARMONIC EQUATION

The determination of the stresses and displacements in a homogeneous isotropic linearly elastic body in a state of plane strain or in a plane stress reduces to solving a boundary-value problem for a system of equations of the two-dimensional theory of elasticity in a plane region G with piecewise-smooth boundary Γ, on the part Γ_1 of which distributed forces are specified, while on the remaining parts Γ_2 the displacements are specified. The region G is assumed to be simply connected, and Γ_1 and Γ_2 may consist of several components (the cases $\Gamma_1 = \Gamma$ or $\Gamma_2 = \Gamma$ are not excluded). If there are no mass forces in G, the use of the Airy stress function $\varphi(x,y)$ (see (2.4)) leads to a boundary-value problem for the biharmonic equation

$$\Delta^2\varphi(x, y) = 0 \quad \text{in } G \tag{2.1}$$

$$\sigma_{nx}|_{\Gamma_1} = X(s), \quad \sigma_{ny}|_{\Gamma_1} = Y(s), \quad u|_{\Gamma_2} = U(s), \quad \upsilon|_{\Gamma_2} = V(s) \tag{2.2}$$

where s is the variable length of the arc of the curve Γ with piecewise-continuous outward normal $n(s)$; $X(s)$, $Y(s)$ and $U(s)$, $V(s)$ are the projections of the distributed forces, specified on Γ_1, and the displacements of the boundary points, specified on Γ_2, onto the axes of a Cartesian system of coordinates, and $\sigma_{nx}|_\Gamma$, $\sigma_{ny}|_\Gamma$ are the projections onto these axes of the internal stresses and boundary points of G

$$\sigma_{nx}|_\Gamma = \frac{d}{ds}\left(\left.\frac{\partial\varphi}{\partial y}\right|_\Gamma\right) \equiv \left.\frac{\partial^2\varphi}{\partial s\partial y}\right|_\Gamma, \quad \sigma_{ny}|_\Gamma = -\frac{d}{ds}\left(\left.\frac{\partial\varphi}{\partial x}\right|_\Gamma\right) \equiv -\left.\frac{\partial^2\varphi}{\partial s\partial x}\right|_\Gamma \tag{2.3}$$

When finding a regular solution (continuous stress and displacement fields in the closed region G) we will assume that everywhere, apart from corner points of the boundary Γ, the functions $X(s)$ and $Y(s)$ are continuous on Γ_1, and $U(s)$ and $V(s)$ are continuously differentiable on Γ_2. At corner points of the boundary special conditions for matching one-sided limit values of the functions $X(s)$, $Y(s)$, $U(s)$, $V(s)$ and their derivatives, which are necessary for a regular solution to exist [9], must be satisfied. If $\Gamma_1 = \Gamma$, the specified boundary forces must satisfy three equations of equilibrium of the region G as a whole. The solution of problem (2.1), (2.2) will be sought in the factor-space $K(G)$ of the space of all functions, of the class $C^4(G) \cap C^2(\bar{G})$, which are biharmonic in G with respect to the subspace of linear functions (the zero stress field (2.4) corresponds to these).

The stresses and displacements at any point of the closed region $\bar{G}$ can be expressed in terms of $\varphi(x, y)$ by the formulae

$$\sigma_x(x,y) = \frac{\partial^2\varphi}{\partial y^2}, \quad \sigma_y(x,y) = \frac{\partial^2\varphi}{\partial x^2}, \quad \tau_{xy}(x,y) = -\frac{\partial^2\varphi}{\partial x\partial y} \tag{2.4}$$

$$\begin{Vmatrix} u(x,y) \\ \upsilon(x,y) \end{Vmatrix} = \begin{Vmatrix} \mathbf{D}_u\varphi \\ \mathbf{D}_\upsilon\varphi \end{Vmatrix} + \begin{Vmatrix} -\omega_0 y + y_0\omega_0 + u_0 \\ \omega_0 x - x_0\omega_0 + \upsilon_0 \end{Vmatrix} \tag{2.5}$$

$$\mathbf{D}_u\varphi = E_*^{-1}\left(-\nu\,\frac{\partial\varphi}{\partial x}(x,y) + \int_{x_0}^{x}\frac{\partial^2\varphi}{\partial y^2}(\xi,y)d\xi - \int_{y_0}^{y}\left(\int_{y_0}^{\xi}\frac{\partial^3\varphi}{\partial x^3}(x_0,t)dt\right)d\xi - \right.$$

$$-2\frac{\partial\varphi}{\partial x}(x_0,y)+y\frac{\partial^2\varphi}{\partial x\partial y}(x_0,y_0)-y_0\frac{\partial^2\varphi}{\partial x\partial y}(x_0,y_0)+(\nu_*+2)\frac{\partial\varphi}{\partial x}(x_0,y_0)\Bigg) \tag{2.6}$$

$$\mathbf{D}_\upsilon\varphi=E_*^{-1}\Bigg(-\nu_*\frac{\partial\varphi}{\partial y}(x,y)+\int_{y_0}^{y}\frac{\partial^2\varphi}{\partial x^2}(x,\xi)d\xi-\int_{x_0}^{x}\left(\int_{x_0}^{\xi}\frac{\partial^3\varphi}{\partial y^3}(t,y_0)dt\right)d\xi-$$

$$-2\frac{\partial\varphi}{\partial y}(x,y_0)+x\frac{\partial^2\varphi}{\partial x\partial y}(x_0,y_0)-x_0\frac{\partial^2\varphi}{\partial x\partial y}(x_0,y_0)+(\nu_*+2)\frac{\partial\varphi}{\partial y}(x_0,y_0)\Bigg) \tag{2.7}$$

where, in the case of plane stress, $E_* = E$ and $\nu_* = \nu$ are the modulus of elasticity and Poisson's ratio of the elastic medium, while in the case of plane strain $E_* = E(1-\nu^2)^{-1}$, is an arbitrarily fixed point of the region G, $\omega(x,y) = 1/2(\partial\upsilon/\partial x - \partial u/\partial y)$ is the angle of rigid rotation at the point (x,y), $u_0 = u(x_0, y_0)$, $\upsilon_0 = \upsilon(x_0,y_0)$, $\omega_0 = \omega(x_0,y_0)$ are constants, which define the displacement of the region G as an absolutely rigid body, and

$$\mathbf{D}=\left\|\begin{matrix}\mathbf{D}_u\\ \mathbf{D}_\upsilon\end{matrix}\right\|$$

is a linear integro-differential operator [7] which maps every biharmonic function $\varphi(x,y) \in K(G)$ into the uniquely defined elastic component $\mathbf{D}\varphi$ of the displacement field (i.e. the vector field in G satisfying the homogeneous two-dimensional Lamé equations) with zero displacements and zero angle of rigid rotation at the point (x_0, y_0).

Formulae (2.5)–(2.7) (derived in [7]) express the displacement field in the simply connected region G in terms of the stress function in explicit form, suitable as a detailed instruction for a computer, and enable the mixed problem of the two-dimensional theory of elasticity to be reduced to a biharmonic boundary-value problem for the stress function. They can be regarded as one more form of the general solution of the homogeneous Lamé equations in a simply connected region.

After the stress function φ has been obtained by solving boundary-value problem (2.1), (2.2), the stresses and displacements at any point of the closed region $\bar{G}$ can be determined, using (2.4)–(2.7).

By changing to the boundary-value problem for the biharmonic equation (which is usual only in case when $\Gamma_1 = \Gamma$, if it is not required to obtain the displacements), one can use the same basis system $\{\varphi_i\}$ to solve problems of the theory of elasticity and of bending of plates.

3. APPROXIMATE SOLUTION

An approximate solution of problem (2.1), (2.2) is constructed in the form

$$\varphi(x,y)=\varphi^N(x,y)=\sum_{-2}^{N}c_i\varphi_i(x,y) \tag{3.1}$$

where $\varphi(x,y)$ $(i = 1, 2, \ldots)$ is a certain system of linearly independent functions, biharmonic in G, from the space $K(G)$. For any N and c_i the function (3.1) satisfies Eq. (2.1) in G. The coefficients c_i, which give the best approximation of the boundary conditions (2.2), are found from the condition for the functional

$$F[\varphi]=\sum_{l=0}^{1}\int_{\Gamma_1}(((\sigma_{nx}|_{\Gamma_1}-X(s))^{(l)})^2+((\sigma_{ny}|_{\Gamma_1}-Y(s))^{(l)})^2)w_l(s)ds+$$

$$+\gamma\sum_{l=0}^{1}\int_{\Gamma_2}(((u|_{\Gamma_2}-U(s))^{(l)})^2+((v|_{\Gamma_2}-V(s))^{(l)})^2)w_l(s)ds \tag{3.2}$$

to be a minimum on K. Here $g^{(l)}(s) = d^lg/ds^l$, the scaling factor $\gamma = E_*^2/|\Gamma|^2$ serves to equalize the physical dimensions and magnitude of the summands, and the weighting functions $w_0(s) > 0$, $w_1(s) \geqslant 0$, piecewise-continuous on Γ, serve to adjust the functional in accordance with the desired properties of the sequence of approximate solutions (for example, $w_1(s)$ may be a piecewise-constant function, which takes the greatest value on those parts of the boundary where a more accurate approximation is desirable, and is equal to zero where the corresponding discrepancy is not required to be small). The problem of minimizing the functional (3.2) on K (or $K(G) \cap C^3(\bar{G})$ when $w_1(s) \not\equiv 0$) is equivalent (see Section 1) to boundary-value problem (2.1), (2.2). Assuming the existence and uniqueness of a regular solution of problem (2.1), (2.2), functional (3.2) has a unique minimum in K—the exact solution of problem (2.1), (2.2).

The displacement field (2.5), that corresponds to the stress function (3.1), is of the form

$$\left\| \begin{matrix} u(x,y) \\ v(x,y) \end{matrix} \right\| = \sum_{-2}^{N} c_i \left\| \begin{matrix} u_i(x,y) \\ v_i(x,y) \end{matrix} \right\| \tag{3.3}$$

$$\left\| \begin{matrix} u_{-2} \\ v_{-2} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|, \quad \left\| \begin{matrix} u_{-1} \\ v_{-1} \end{matrix} \right\| = \left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\|, \quad \left\| \begin{matrix} u_0 \\ v_0 \end{matrix} \right\| = \left\| \begin{matrix} y \\ -x \end{matrix} \right\|, \quad \left\| \begin{matrix} u_i \\ v_i \end{matrix} \right\| = \left\| \begin{matrix} \mathbf{D}_u \varphi_i \\ \mathbf{D}_v \varphi_i \end{matrix} \right\| \equiv \mathbf{D}\varphi_i, \quad i \geqslant 1 \tag{3.4}$$

The corresponding boundary stresses can be calculated from (2.3)

$$\sigma_{nx}|_\Gamma = \sum_{-2}^{N} c_i \frac{\partial^2 \varphi_i}{\partial s \partial y}\bigg|_\Gamma, \quad \sigma_{ny}|_\Gamma = -\sum_{-2}^{N} c_i \frac{\partial^2 \varphi_i}{\partial s \partial x}\bigg|_\Gamma \tag{3.5}$$

where $\varphi_{-2} \equiv \varphi_{-1} \equiv \varphi_{-0} \equiv 0$. Substituting (3.3) and (3.5) into (3.2) we can convert the functional F into a quadratic function $F(c_{-2}, \ldots, c_N)$ on $\mathbb{R}^{N+3}$. The necessary conditions for it to have an extremum $\partial F/\partial c_k = 0$ lead to a system of $N + 3$ linear equations for determining the coefficients c_i, which give the best approximation of the boundary conditions (2.2)

$$\sum_{i=-2}^{N} p_{ki} c_i = q_k, \quad k = -2, -1, \ldots, N \tag{3.6}$$

$$p_{ki} = \sum_{l=0}^{1} \int_{\Gamma_1} \left(\frac{\partial^2 \varphi_k}{\partial s \partial y}\bigg|_{\Gamma_1}^{(l)} \frac{\partial^2 \varphi_i}{\partial s \partial y}\bigg|_{\Gamma_1}^{(l)} + \frac{\partial^2 \varphi_k}{\partial s \partial x}\bigg|_{\Gamma_1}^{(l)} \frac{\partial^2 \varphi_i}{\partial s \partial x}\bigg|_{\Gamma_1}^{(l)} \right) w_l(s) ds +$$

$$+\gamma \sum_{l=0}^{1} \int_{\Gamma_2} (u_k|_{\Gamma_2}^{(l)} u_i|_{\Gamma_2}^{(l)} + v_k|_{\Gamma_2}^{(l)} v_i|_{\Gamma_2}^{(l)}) w_l(s) ds \tag{3.7}$$

$$q_k = \sum_{l=0}^{1} \int_{\Gamma_1} \left(X^{(l)}(s) \frac{\partial^2 \varphi_k}{\partial s \partial y}\bigg|_{\Gamma_1}^{(l)} - Y^{(l)}(s) \frac{\partial^2 \varphi_k}{\partial s \partial x}\bigg|_{\Gamma_1}^{(l)} \right) w_l(s) ds +$$

$$+\gamma \sum_{l=0}^{1} \int_{\Gamma_2} (U^{(l)}(s) u_k|_{\Gamma_2}^{(l)} + V^{(l)}(s) v_k|_{\Gamma_2}^{(l)}) w_l(s) ds \tag{3.8}$$

The matrix $\|p_{ki}\|$ is symmetrical and positive definite (like a matrix of positive quadratic form). Hence, for any N, system (3.6) has a unique solution, and the non-negative quadratic function $F(c_{-2}, \ldots, c_N)$ is a minimum for this solution. When $\Gamma_1 = \Gamma$, (3.6) becomes a system of N equations for c_i, $i > 0$ ($p_{ki} = 0$, $q_k = 0$ when $k \leqslant 0$ or $i \leqslant 0$), while to obtain the coefficients c_i, $i < 0$, which define the rigid displacement of the region $\bar{G}$, it is necessary to specify a statistically determinable system of supports for $\bar{G}$ (for example, consisting of three supports which do not allow displacement of three fixed points along their axes [7]). The values of c_i obtained for each N from the system (3.6) define the sequence of approximate solutions φ^N (3.1).

Each of the above-mentioned properties of the family of basis functions φ_i plays an important role: the condition $\varphi_i \in K(G)$ ensures that the approximate solution (3.1) belongs to the space $K(G)$, in particular, that it is biharmonic in G; the linear independence of $\{\varphi_i\}$ guarantees the existence and uniqueness of the solution of system (3.6) for any N; the F-completeness of $\{\varphi_i\}$ ensures the construction of a minimizing sequence of the functional (3.2), i.e. the convergence of the boundary residual to zero and the convergence of the sequence of approximations (3.1) to the exact solution of problem (2.1), (2.2) as $N \to \infty$.

If G is a simply connected region, any basis in the linear space of all biharmonic polynomials satisfies all these requirements on the family φ_i. Its F-completeness is a consequence of the following assertion: any function $\varphi(x, y) \in C^{m+2}(\bar{G})$, $m \geqslant 1$, that is biharmonic in a bounded simply connected plane region G, is the limit of a sequence of biharmonic polynomials which converges in the norm $C^m(\bar{G})$ (i.e. which converges uniformly in $\bar{G}$ together with the sequences of derivatives up to order m).

Proof. By Goursat's theorem any functions $f(z)$, $g(z)$, $z = x + iy$, holomorphic in the region G, generates a biharmonic function $\varphi(x, y)$, $\mathrm{Re}(\bar{z}f(z) + g(z))$, and if G is simply connected, every function that is biharmonic in G can be represented in this form [11]. Here $f(z) = \frac{1}{4}\int \Phi(z)dz$, where $\Phi(z)$ is such a holomorphic function in G, that $\mathrm{Re}\,\Phi = \Delta\varphi$, while $\mathrm{Re}\, g = \Delta\varphi - x\mathrm{Re}\, f - y\mathrm{Im}\, f$. Thus, it follows from $\varphi \in C^{m+2}(\bar{G})$ that $\mathrm{Re}\,\Phi = \Delta\varphi \in C^m(\bar{G})$. Hence

$\operatorname{Im}\Phi \in C^m(\bar{G})$ by the Cauchy–Riemann equations $\Phi \in C^m(\bar{G})$, and therefore $f \in C^{m+1}(\bar{G})$, $\operatorname{Re} g \in C^m(\bar{G})$, and by virtue of the Cauchy–Riemann equations $\operatorname{Im} g \in C^m(\bar{G})$, i.e. $g \in C^m(\bar{G})$.

By Mergelyan's theorem [12] (any function of the complex variable z, continuous in the compactum $E \subset \mathbb{C}$ and holomorphic in its interior, is the limit of a sequence of polynomials of z, which converges uniformly in E, if and only if the supplement $\mathbb{C}\backslash E$ is connected), sequences of polynomials of z exist which converge uniformly to $f^{(m)}(z)$ and $g^{(m)}(z)$ in the compactum $\bar{G}$. By integrating them m times we obtain sequences of polynomials $f_n(z)$ and $g_n(z)$ such that $\{f_n^{(l)}(z)\}$ and $\{g_n^{(l)}(z)\}$ converge uniformly in $\bar{G}$ to $f^{(l)}(z)$ and $g^{(l)}(z)$ provided $l \leqslant m$ (consequently, $\operatorname{Re} f_n \to \operatorname{Re} f$, $\operatorname{Im} f_n \to \operatorname{Im} f$ in the norm of $C^m(\bar{G})$). Then, the sequence of biharmonic (by Goursat's theorem) polynomials $p_n(x,y) = \operatorname{Re}(\bar{z} f_n(z) + g_n(z)) = x \operatorname{Re} f_n + y \operatorname{Im} f_n + \operatorname{Re} g_n$ converges to $\varphi(x,y)$ in the norm of $C^m(\bar{G})$: for $\alpha, \beta \geqslant 0$ such that $\alpha + \beta \leqslant m$, the sequences

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}(p_n - \varphi) = \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\operatorname{Re}(\bar{z}(f_n - f) + g_n - g) =$$

$$= \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}(x \operatorname{Re}(f_n - f) + y \operatorname{Im}(f_n - f) + \operatorname{Re}(g_n - g))$$

converge uniformly in $\bar{G}$ to zero, since $\{\operatorname{Re}(f_n - f)\}$, $\{\operatorname{Im}(f_n - f)\}$, $\{\operatorname{Re}(g_n - g)\}$ converges to zero in $C^m(\bar{G})$ while $|x|, |y| \leqslant C$ (G is bounded).

This assertion can be extended to multiply connected regions if we replace the polynomials by rational biharmonic functions having no poles in $\bar{G}$. Similar theorems on the uniform approximation of any solution by polynomial or rational solutions also hold for Laplace's equation [8] and the one-dimensional wave equation.

Since the homogeneous components of any biharmonic polynomial are biharmonic, and the dimension of the spaces P_m of homogeneous biharmonic polynomials of degree $m \geqslant 3$ is equal to 4, the basis of the space of all biharmonic polynomials can be obtained by combining into one chain the bases $\{p_{mj}, j = 1, 2, 3, 4\}$ of the subspaces P_m, constructed in [7]

$$\varphi_{-2} = 1, \quad \varphi_{-1} = x, \quad \varphi_0 = y, \quad \varphi_1 = x^2, \quad \varphi_2 = 2xy, \quad \varphi_3 = y^2, \quad \varphi_i = p_{mj}(x,y), \quad i > 3 \qquad (3.9)$$

$$p_{m1} = \sum_{l=0}^{[m/2]} (-1)^l (1-l) \frac{x^{m-2l}}{(m-2l)!} \frac{y^{2l}}{(2l)!}, \quad p_{m3} = -\sum_{l=0}^{[m/2]} (-1)^l l \frac{x^{m-2l}}{(m-2l)!} \frac{y^{2l}}{(2l)!}$$

$$p_{m2} = \sum_{l=0}^{L} (-1)^l (1-l) \frac{x^{m-2l-1}}{(m-2l-1)!} \frac{y^{2l+1}}{(2l+1)!}, \quad p_{m4} = -\sum_{l=0}^{L} (-1)^l l \frac{x^{m-2l-1}}{(m-2l-1)!} \frac{y^{2l+1}}{(2l+1)!}$$

$$m = \left[\frac{i}{4}\right] + 2, \quad j = i - 4\left[\frac{i}{4}\right] + 1, \quad L = \left[\frac{m-1}{2}\right].$$

With this choice of φ_i the stress function (3.1) turns out to be a biharmonic polynomial of order $[N/4] + 2$, and analysis of the structure of the operators $\mathbf{D}_u$, $\mathbf{D}_v$ defined by (2.6) and (2.7) for $x_0 = y_0 = 0$ shows that the vector field $\mathbf{D}\varphi_i$ in (3.3) and (3.4) is a homogeneous polynomial solution of the Lamé equations, and the linear independence of the set $\{\mathbf{D}\varphi_i\}$ follows from the linear independence of $\{\varphi_i\}$. Hence, the basis system of biharmonic polynomials φ_i generates a complete system (3.4) of linearly independent polynomial solutions of the two-dimensional Lamé equations. After substituting (3.9) into (3.4) and evaluating the derivatives and integrals in (2.6) and (2.7) (for $x_0 = y_0 = 0$), expressions were obtained in [7] for the coefficients of the polynomials u_i and v_i in terms of the known coefficients of polynomials (3.9).

4. NUMERICAL IMPLEMENTATION OF THE METHOD

The polynomial basis functions φ_i possess at least two attractive features: (1) they are not linked to any specific region G and ensures the convergence of the approximate solutions to the exact ones for any simply connected region; (2) they are convenient to use in calculations. For example, if the parametric equations Γ, $w_f(s)$ and $g(s)$ are piecewise-polynomial functions (by virtue of the approximative properties of these functions this class of problems covers all the ones arising in engineering), all the integrals in (3.7), (3.8) can be evaluated analytically (using packets of symbol calculations), and algebraic formulae are obtained for p_{ki} and q_k. In this case we can eliminate the use of numerical-integration procedures—the main source of computational errors and processor costs—and considerably increase the accuracy and efficiency of the algorithm.

In general, the matrix of system (3.6) is not a sparse matrix, as in all variational methods based on global approximation. However, an analysis of (3.7) and (3.8) reveals considerable simplifications in the structure of system (3.6) provided the region G, the basis functions φ_i and the boundary conditions possess symmetry properties. The sufficient conditions for the coefficients p_{ki} and q_k to vanish were obtained in [7]. In particular, it was established that if the parts Γ_1 and Γ_2 of the boundary G are symmetrical about both coordinate axes, three-quarters of the total number of elements p_{ki} will be zero and arranged so that system (3.6) splits up into four independent subsystems. Then, depending on the type of symmetry of the boundary conditions (2.2), one, two or three of these will certainly have a zero solution.

The use of polynomial basis functions and orientation towards the analytic (symbolic) inputting and processing of data enabled the author to develop [7] a general computational algorithm and software package for a highly accurate computation of stresses and displacements in a rectangular elastic region which are caused by displacements specified on a part of its boundary and piecewise-continuous distributed forces, and concentrated forces applied to the remaining part. The load, including concentrated forces and distributed forces with discontinuities of the first kind, is reduced to continuous boundary conditions of the form (2.2) by preliminary automatic separation of the analytically constructed stress and displacement fields with a special type of singularities at specified boundary points (they are not an additional source of error since they satisfy all the equations of the theory of elasticity) [7, 10]. This preliminary regularization of the boundary conditions improves the computational properties of system (3.6) and enables N to be reduced.

An important feature of this approach and the program which implements it is the possibility of simple monitoring of the reliability and degree of accuracy of the numerical results obtained: since the approximate solution satisfies all the equations of the theory of elasticity identically, it is sufficient to check that the deviations of the computed boundary values of the stresses and displacements from the given ones are small, and do not exceed the errors in specifying the boundary conditions (in fact, the exact solution of the problem is obtained with only small changes in the boundary conditions).

The program developed was used, during the six years of its existence, to solve several dozens of problems of the theory of elasticity with a variety of boundary conditions [7, 10], in particular, those shown in Figs 1 and 2. A detailed analysis of the approximate solutions of various problems obtained for different N showed that for $N = 160$ (using polynomials (3.9) up to order 40) the relative error of the stresses and displacements calculated at 200–400 boundary points did not exceed 10^{-4}–10^{-7}, and even for $N \approx 60$ an accuracy of 1% was achieved in the majority of cases, i.e. the solutions obtained are practically indistinguishable from the exact ones. Because of the rapid convergence (small values of N in practical calculations), the negative effect of the filling of the matrix of system (3.6) and the degradation of its conditionality as N increases is unable to develop in practice (the common knowledge concerning the "almost linear dependence" of systems of polynomial basis functions is not confirmed for biharmonic polynomials (3.9)).

(Bi)harmonic polynomials are suitable for solving problems in hydromechanics, electrostatics, the St Venant theory for torsion and bending of prismatic bodies, the bending of membranes and plates, and the three-dimensional theory of elasticity.

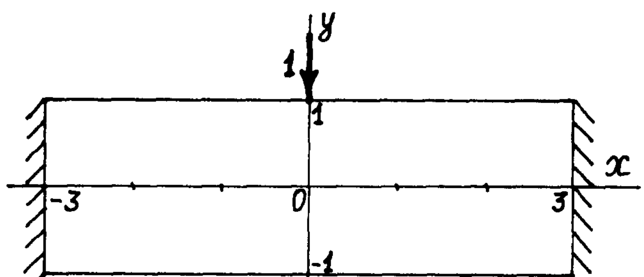

Fig. 1.

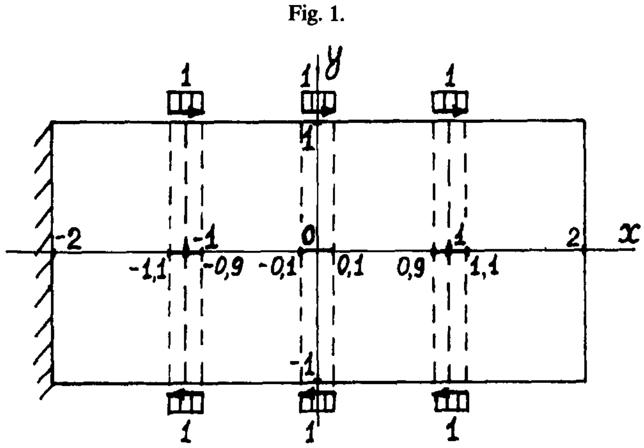

Fig. 2.

5. EXAMPLES

In the problem shown in Fig. 1, the sides $x = \pm 3$ are rigidly clamped. Graphs of the stresses in the sections $x =$ const and $y =$ const are shown in Figs 3–5 (in view of the symmetry $\sigma_x(-x, y) = \sigma_x(x, y)$, $\tau_{xy}(-x, y) = -\tau_{xy}(x, y)$ and hence only half the graphs are shown in Figs 3 and 4 for $x \geqslant 0$). It is useful to compare them with the stresses calculated from the formulae for the resistance of materials, denoted by the dashed curves. The concentration of the stresses in the region of the clamped sides and in the neighbourhood of the point where the concentrated forces are applied is of interest.

In the problem shown in Fig. 2, the shear boundary forces are distributed uniformly over six segments of length 0.2. The shear stresses produced by these are shown in Figs 6 and 7 $\tau_{xy}(x, -y) = -\tau_{xy}(x, y)$). At points of discontinuity of the boundary forces the stress field $\tau_{xy}(x, y)$ has a discontinuity of the first kind, while $\sigma_x(x, y)$ has the logarithmic singularity. The maxima of the $\tau_{xy}(y)$ curve in Fig. 6, as in Fig. 5, are situated in the region of the outer fibres and not on the neutral axis.

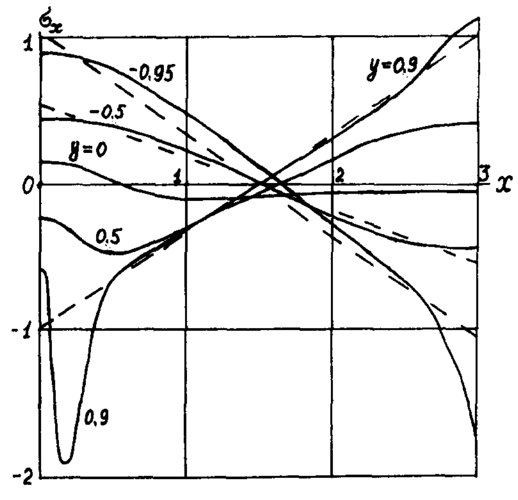

Fig. 3.

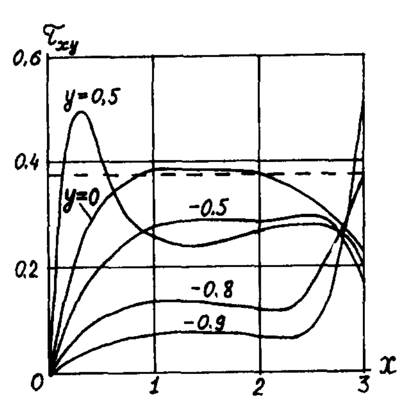

Fig. 4.

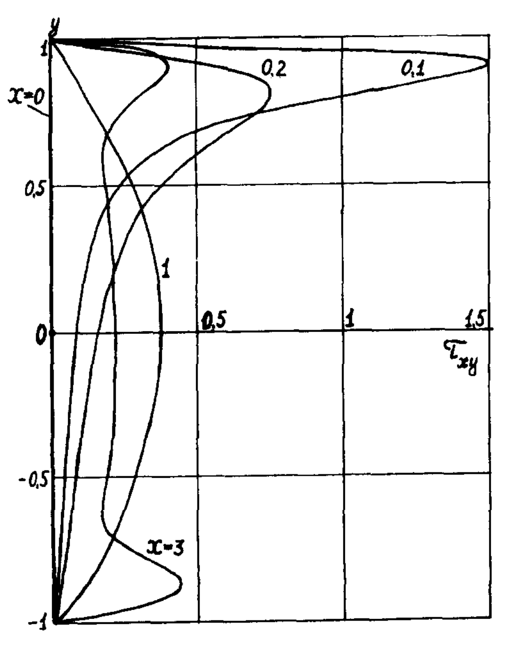

Fig. 5.

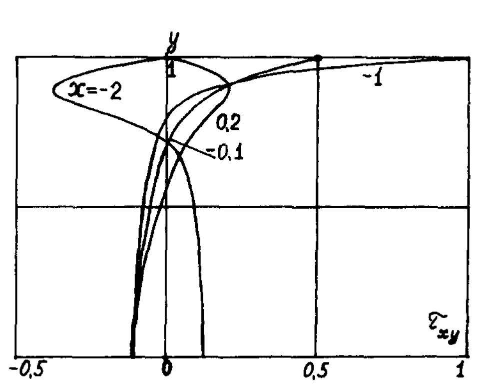

Fig. 6.

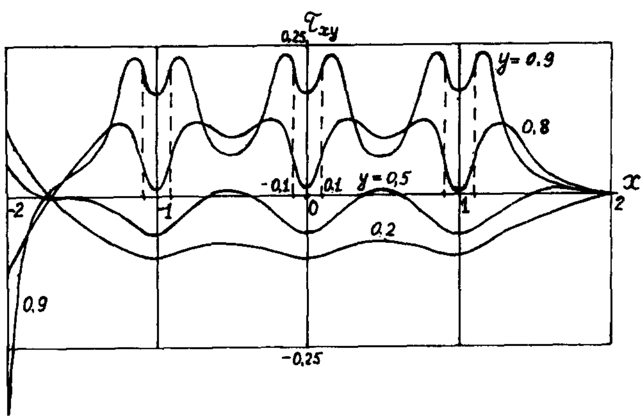

Fig. 7.

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